We have $(\delta_{ab} + \delta g_{ab}(\sigma))e^{2\omega(\sigma)} = \delta_{ab}$ and $\delta g_{ab} = -\partial_a v_b - \partial_b v_a$. These conditions become:

$$\partial_a v_b = -\partial_b v_a, \quad a \neq b \tag{1}$$
$$\partial_a v_a = k(\sigma), \quad \forall a \tag{2}$$

where $k(\sigma)$ is an arbitrary function of the coordinates. Expand v_a and k in Taylor series (summing over repeated indices):

$$v_a = c_a + c_{ab}\sigma_b + c_{abc}\sigma_b\sigma_c + \cdots \tag{3}$$
$$k = k_0 + k_a\sigma_a + k_{ab}\sigma_a\sigma_b + \cdots \tag{4}$$

Condition (1) becomes

$$\begin{aligned} c_{ab} &= -c_{ba}, \quad a \neq b \\ c_{abc} &= -c_{bac}, \quad a \neq b \\ c_{abcd} &= -c_{bacd}, \quad a \neq b \\ &\cdots \end{aligned} \tag{5}$$

Condition (2) becomes

$$\begin{aligned} c_{aa} &= k_0, \quad \forall a \\ 2c_{aac} &= k_c, \quad \forall a \\ 3c_{aacd} &= k_{cd}, \quad \forall a \\ &\cdots \end{aligned} \tag{6}$$

Now we count the available degrees of freedom given these constraints. $c_{(n)}$ refers to the $c_{abc\cdots}$ with n indices. It is totally symmetric in all but the first index. A totally symmetric m-tensor in d dimensions has $\begin{pmatrix} d+m-1 \\ m \end{pmatrix}$ components, so $c_{(n)}$ has

$$d\begin{pmatrix} d+n-2 \\ n-1 \end{pmatrix} \tag{7}$$

components. Conditions (5) are antisymmetric conditions on the first two indices, and there are $\begin{pmatrix} d+n-3 \\ n-2 \end{pmatrix}$ equations from the remaining $n-2$ totally symmetric indices, so they take away a total of

$$\frac{d(d-1)}{2}\begin{pmatrix} d+n-3 \\ n-2 \end{pmatrix} \tag{8}$$

degrees of freedom for each n. Similarly, conditions (6) take away

$$d\begin{pmatrix} d+n-3 \\ n-2 \end{pmatrix} \tag{9}$$

degrees of freedom. Finally, we are also free to choose the components $k_a, k_{ab}, \cdots$, and there are

$$\begin{pmatrix} d+n-3 \\ n-2 \end{pmatrix}, \tag{10}$$

since the k's are totally symmetric. Combining (7), (8), (9), (10), we arrive at

$$D_n = \left(\frac{d(d+n-2)}{n-1} - \frac{d(d+1)}{2} + 1\right)\begin{pmatrix} d+n-3 \\ n-2 \end{pmatrix} \tag{11}$$

degrees of freedom for each $n \geq 2$. We also have $D_1 = d$ from the constant term c_a. Now we sum over $n = 1, 2, \cdots$, keeping in mind D_n can become zero or negative, indicating no solutions. We have

$$D_1 = d \tag{12}$$

$$D_2 = \frac{d(d-1)}{2} + 1 \tag{13}$$

$$D_3 = d. \tag{14}$$

For $d = 2$, we then have $D_n = 2$ for $n \geq 4$, while for $d > 2$, we have $D_n \leq 0$ for $n \geq 4$. This can be seen by solving

$$\left(\frac{d(d+n-2)}{n-1} - \frac{d(d+1)}{2} + 1 \right) = 0 \tag{15}$$

for d. For $n \geq 4$ this is concave, so the positive solution gives the dimension below which $D_n > 0$. We get

$$d = \frac{1 + \sqrt{1 + 8\left(\frac{n-1}{n-3}\right)}}{2}, \tag{16}$$

which approaches 2 from above as $n \to \infty$. Thus, we have a total of $(d+1)(d+2)/2$ solutions for $d > 2$ and an infinite number for $d = 2$.